On unitary ϱ-dilations of operators

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Using the notation of [1], $\mathscr{C}_\varrho$ $(\varrho \geqq 0)$ will denote the class of those (bounded, linear) operators T in a Hilbert space $\mathfrak{H}$, for which

$$T^n h = \varrho P U^n h \qquad (h \in \mathfrak{H};\, n = 1, 2, \ldots)$$

holds, where U is a unitary operator in some Hilbert space $\mathfrak{K}$, containing $\mathfrak{H}$ as a subspace, and P denotes the projection of $\mathfrak{H}$ onto $\mathfrak{K}$.

The following theorem was proved in [1].

Theorem A. *In the case* $\varrho > 0$, $T \in \mathscr{C}_\varrho$ *if and only if*

$$(\mathrm{I}_\varrho) \qquad \|h\|^2 - 2\left(1 - \frac{1}{\varrho}\right) \mathrm{Re}\, (zTh, h) + \left(1 - \frac{2}{\varrho}\right) \|zTh\|^2 \geqq 0 \text{ for } h \in \mathfrak{H},\ |z| \leqq 1;$$

(II) *the spectrum of T lies in the closed unit disc.*

The purpose of this paper is to study the monotonity properties of $\mathscr{C}_\varrho$ as a function of ϱ. Meanwhile we shall give a simple necessary and sufficient condition for a normal T to belong to $\mathscr{C}_\varrho$.

We start with the following

Lemma 1. *$\mathscr{C}_\varrho$ is a non-decreasing function of ϱ in the sense that $\mathscr{C}_{\varrho_1} \subset \mathscr{C}_{\varrho_2}$ if $0 \leqq \varrho_1 < \varrho_2$.*

This lemma was already proved in [1]. Here we give another proof of it as follows.

Proof. The definition of $\mathscr{C}_0$ shows that $T \in \mathscr{C}_0$ if and only if $T = O$ (the zero operator). According to Theorem A we have $O \in \mathscr{C}_\varrho$ for every $\varrho > 0$. This implies our lemma in the case $\varrho_1 = 0$.

Now let $\varrho_1 > 0$, and set

$$F_{z,h}(\varrho) = \varrho \|h\|^2 - 2(\varrho - 1)\, \mathrm{Re}\, (z\, Th, h) + (\varrho - 2)\|zTh\|^2 =$$
$$= \|h\|^2 - \|zTh\|^2 + (\varrho - 1)\|(I - zT)h\|^2 \quad (\varrho > 0,\ |z| \leqq 1,\ h \in \mathfrak{H}).$$

(I_ϱ) holds if and only if $F_{z,h}(\varrho) \geqq 0$ whenever $|z| \leqq 1$ and $h \in \mathfrak{H}$. Now let $T \in \mathscr{C}_{\varrho_1}$ and $\varrho_2 > \varrho_1$. In this case (II) holds, and $F_{z,h}(\varrho_1) \geqq 0$ $(|z| \leqq 1, h \in \mathfrak{H})$. $F_{z,h}(\varrho)$ is a monotone non-decreasing function of ϱ, consequently we also have $F_{z,h}(\varrho_2) \geqq 0$. This implies $T \in \mathscr{C}_{\varrho_2}$.

Theorem 1. *In the case that T is normal, a necessary and sufficient condition for $T \in \mathscr{C}_\varrho$ is*

$$\|T\| \leqq \begin{cases} \dfrac{\varrho}{2-\varrho}, & \text{if} \quad 0 \leqq \varrho \leqq 1, \\ 1, & \text{if} \quad \varrho > 1. \end{cases}$$

Proof. In the case that $\varrho = 0$ our statement is trivial.

Let $0 < \varrho < 2$. In this case (I_ϱ) is equivalent with

$$\frac{\varrho}{\varrho-2}\|h\|^2 - 2\frac{\varrho-1}{\varrho-2}\operatorname{Re}(zTh, h) + \|zTh\|^2 \leqq 0 \qquad (h \in \mathfrak{H}, \quad |z| \leqq 1).$$

This latter relation holds if and only if

$$\left\|\left(\frac{\varrho-1}{\varrho-2}I - zT\right)h\right\|^2 + \frac{\varrho}{\varrho-2}\|h\|^2 \leqq \left(\frac{\varrho-1}{\varrho-2}\right)^2\|h\|^2 \qquad (h \in \mathfrak{H}, \quad |z| \leqq 1)$$

or, equivalently,

$$\left\|\left(\frac{1-\varrho}{2-\varrho}I - zT\right)h\right\| \leqq \frac{1}{2-\varrho}\|h\| \qquad (h \in \mathfrak{H}, \quad |z| \leqq 1),$$

i. e.

$$(\mathrm{I}'_\varrho) \qquad \sup_{|z| \leqq 1}\left\|\frac{1-\varrho}{2-\varrho}I + zT\right\| \leqq \frac{1}{2-\varrho}.$$

So we see that

(1) (I_ϱ) *and* (I'_ϱ) *are equivalent for* $0 < \varrho < 2$.

Moreover,

$$\left\|\frac{1-\varrho}{2-\varrho}I + zT\right\| \leqq \left\|\frac{1-\varrho}{2-\varrho}I\right\| + \|zT\|,$$

consequently

$$(2) \qquad \sup_{|z| \leqq 1}\left\|\frac{1-\varrho}{2-\varrho}I + zT\right\| \leqq \frac{1-\varrho}{2-\varrho} + \|T\|.$$

Now let T be normal. Then the spectrum of T contains a complex number of modulus $\|T\|$, say $\zeta^{-1}\|T\|$ and this is an approximative proper value of T, i.e. for every $\varepsilon > 0$ there exists an element h_ε of $\mathfrak{H}$ such that $\|h_\varepsilon\| = 1$ and

$$\|(\|T\|I - \zeta T)h_\varepsilon\| < \varepsilon.$$

Using this fact we have

$$\left\|\left(\frac{1-\varrho}{2-\varrho}I + \zeta T\right)h_\varepsilon\right\| \geqq \left\|\left(\frac{1-\varrho}{2-\varrho}I + \|T\|I\right)h_\varepsilon\right\| - \|(\|T\|I - \zeta T)h_\varepsilon\| > \frac{1-\varrho}{2-\varrho} + \|T\| - \varepsilon.$$

This is true for every $\varepsilon > 0$, consequently

$$\sup_{|z| \leqq 1}\left\|\frac{1-\varrho}{2-\varrho}I + zT\right\| \geqq \frac{1-\varrho}{2-\varrho} + \|T\|.$$

Our latter relation and (2) show that

$$\sup_{|z|\leqq 1}\left\|\frac{1-\varrho}{2-\varrho}I+zT\right\|=\frac{1-\varrho}{2-\varrho}+\|T\|.$$

This implies that, in the case that T is normal, (I'_ϱ) is equivalent to

$$\frac{1-\varrho}{2-\varrho}+\|T\|\leqq\frac{1}{2-\varrho}$$

or to

$$(\mathrm{I}''_\varrho)\qquad \|T\|\leqq\frac{\varrho}{2-\varrho}.$$

Using (1) we have:

(I_ϱ) and (I''_ϱ) are equivalent for $0<\varrho<2$ if T is normal.

Now let $0<\varrho\leqq 1$. In this case (I''_ϱ) implies (II). Moreover, using Theorem A, we have: In the case $0<\varrho\leqq 1$ and T is normal, $T\in\mathscr{C}_\varrho$ if and only if (I''_ϱ) holds.

Now, for normal T, (II) is equivalent to the condition $\|T\|\leqq 1$. Thus, by Lemma 1, if T is normal, we have $T\in\mathscr{C}_\varrho$ for $\varrho>1$ if and only if $\|T\|\leqq 1$.

So we finished the proof.

For $0\leqq\varrho\leqq 1$, $\dfrac{\varrho}{2-\varrho}$ is strictly increasing function of ϱ. Thus, by Theorem 1 and Lemma 1, $\mathscr{C}_\varrho$ is a strictly increasing function of ϱ for $0\leqq\varrho\leqq 1$.

If $\dim\mathfrak{H}=1$ then there exist only operators of multiplication by complex numbers, and these are normal. In this case, Theorem 1 shows the monotonity properties of $\mathscr{C}_\varrho$.

Theorem 2. *If $\dim\mathfrak{H}\geqq 2$ then $\mathscr{C}_\varrho$ is a strictly increasing function of ϱ in the sense that*

$$\mathscr{C}_{\varrho_1}\subset\mathscr{C}_{\varrho_2}\quad\textit{and}\quad\mathscr{C}_{\varrho_1}\neq\mathscr{C}_{\varrho_2}\quad\textit{if}\quad 0\leqq\varrho_1<\varrho_2.$$

Proof. For arbitrary $\varrho\geqq 0$ we shall construct an operator T_ϱ such that $T_\varrho\in\mathscr{C}_\varrho$ and $\|T_\varrho\|=\varrho$. This T_ϱ does not belong to $\mathscr{C}_\sigma$ if $0\leqq\sigma<\varrho$. This fact and Lemma 1 will prove our theorem.

Let

$$(4)\qquad \{\varphi_1,\varphi_2,\psi_\nu\ (\nu\in\Omega)\}$$

be an orthonormal basis in $\mathfrak{H}$. We define T_ϱ by

$$(5)\qquad T_\varrho\varphi_1=\varrho\varphi_2,\quad T_\varrho\varphi_2=0,\quad T_\varrho\psi_\nu=0\qquad(\nu\in\Omega).$$

Evidently, $\|T_\varrho\|=\varrho$ and

$$(6)\qquad T_\varrho^n=O\qquad(n=2,3,\ldots).$$

Let us construct an orthonormal system

$$(7)\qquad \{\varphi'_m\ (m=0,\pm1,\ldots),\ \psi'_{\nu,m}\ (\nu\in\Omega,\ m=0,\pm1,\ldots)\}$$

and identify φ_k with φ'_k $(k=1,2)$ and ψ_ν with $\psi'_{\nu,0}$. So the Hilbert space $\mathfrak{K}$ spanned by the system (7) will contain $\mathfrak{H}$ as a subspace. Let P be the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$ and define the linear operator U on $\mathfrak{K}$ by

$$U\varphi'_m = \varphi'_{m+1}, \quad U\psi'_{\nu,m} = \psi'_{\nu,m+1} \qquad (\nu\in\Omega; \quad m=0, \pm 1, \ldots).$$

Evidently, U is unitary and we have

$$\varrho PU\varphi_1 = \varrho P\varphi_2 = \varrho\varphi_2, \quad \varrho PU\varphi_2 = \varrho P\varphi'_3 = 0, \quad \varrho PU\psi_\nu = \varrho P\psi'_{\nu,1} = 0.$$

Consequently, by (5),

$$Th = \varrho PUh \qquad (h\in\mathfrak{H}). \tag{8}$$

For $m \geqq 2$ we get

$$\varrho PU^m\varphi_k = \varrho P\varphi'_{k+m} = 0, \quad \varrho PU^m\psi_\nu = \varrho P\psi'_{\nu,m} = 0 \qquad (k=1,2; \quad \nu\in\Omega),$$

consequently,

$$T^n h = 0 \qquad (h\in\mathfrak{H}, \quad n\geqq 2).$$

Thus, by (6) and (8), $T_\varrho \in \mathscr{C}_\varrho$.

So the proof is complete.

Reference

[1] B. Sz.-Nagy and C. Foiaş, On certain classes of power-bounded operators in Hilbert space, *Acta Sci. Math.*, **27** (1966), 17—25.

(Received June 1, 1966)